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Equivalences of the Large Deviation Principle for Gibbs Measures and Critical Balance in the Ising Model

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In this paper we obtain the equivalence of the large deviation principle for Gibbs measures with and without an external field. For the Ising model, the equivalence allows us to study the result of competing influences of a positive external field h and a negative boundary condition in the cube $\Lambda(B/h)$ as $h \searrow 0$ for various B. We find a critical balance at a value B_0 of B.

KEY WORDS: Ising model; Gibbs measures; large-deviation principle.

1. INTRODUCTION

Consider the Ising model on Z^d with positive external field h, in a cube $\Lambda(B/h)$ with side length B/h, and negative boundary condition. Interesting results about the competing influences of a positive external field h and a negative boundary condition as $h \searrow 0$ have been obtained by Martirosyan,[6] Schonmann,[11] and Schonmann and Shlosman.[12,13] Schonmann and Shlosman[13] prove that, for $D=2$, if one looks at the Gibbs measure in the cube $\Lambda(B/h)$ with external field h and negative boundary condition, then there exists a $B_0 > 0$ such that when $0 < B < B_0$ one sees only the $(-)$ phase as $h \searrow 0$, but when $B > B_0$, one sees a $(+)$ phase inside the cube as $h \searrow 0$. Their tools are associated with the large-deviation principle in the case of no external field. This suggests looking at the relationship between the large deviation principles with and without external field. In this paper, we will prove that there is equivalence between

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the large-deviation principles for random fields (not only for the Ising model) with and without the external field for all dimensions $d \geqslant 1$ (Theorem 2.1). We apply this result to the d-dimensional Ising model to prove that, for $d \geqslant 2$ and T subcritical, there exists a critical value B_0 such that different phenomena occur for $B < B_0$ and $B > B_0$ in terms of the average spin (see Section 3 for more details). When $d = 2$, we verify that the critical value B_0 coincides with the one obtained by Schonmann and Shlosman.[13] While studying the competing influences of the positive external field h and the negative boundary condition in $\Lambda(B/h)$ for $d = 2$, one obtains a large-deviation principle which complements some results of Schonmann and Shlosman.[13] We briefly discuss the case of $B = B_0$.

We use the compactness method in large deviation theory developed by O'Brien and Vervaat.[8]

2. THE LARGE DEVIATION RESULT

Let $E \subset R$ be a finite set. One can take E to be more general, for example, compact, but we restrict ourselves for simplicity. Let $\Omega = E^{Z^d}$ be the configuration space. For all finite sets $\Lambda \subset Z^d$, define $\Omega(\Lambda) = E^{\Lambda}$. Typical configurations in Ω or $\Omega(\Lambda)$ are denoted by $\sigma, \eta, \ldots$. Denote the value of σ at $x \in Z^d$ by σ_x. Let $H_{\Lambda, \eta}(\sigma)$ be an energy function in the set Λ, with boundary condition η, evaluated on the configuration σ, for example, (3.1). Then

$$H_{\Lambda, \eta, s}(\sigma) = \frac{1}{2} H_{\Lambda, \eta}(\sigma) - \frac{s}{2} \sum_{x \in \Lambda} \sigma_x \tag{2.1}$$

is called the energy with the external field s. We include the factor 1/2 to agree with the notation of Schonmann and Shlosman.[13] Let $\beta = 1/T$, where T is the temperature. Then the Gibbs measure in Λ with boundary condition η, temperature T, and external field s is defined by

$$\mu_{\Lambda, \eta, T, s}(\sigma) := \exp(-\beta H_{\Lambda, \eta, s}(\sigma)) / Z_{\Lambda, \eta, T, s}, \qquad \forall \sigma \in \Omega(\Lambda) \tag{2.2}$$

where $Z_{\Lambda, \eta, T, s}$ is the normalizing constant given by

$$Z_{\Lambda, \eta, T, s} := \sum_{\sigma \in \Omega(\Lambda)} \exp(-\beta H_{\Lambda, \eta, s}(\sigma)) \tag{2.3}$$

Expectation with respect to $\mu_{\Lambda, \eta, T, s}$ is denoted by $E_{\Lambda, \eta, T, s}$. Note that $\mu_{\Lambda, \eta, T, 0}$ is the Gibbs measure in the absence of an external field.

For all $h>0$, let $\Lambda(1/h)$ be the cube in Z^d with side length $1/h$ and centered at the origin. The average of σ in $\Lambda(1/h)$, called the average spin, is defined by

$$X_{\Lambda(1/h)}(\sigma):=(1/h)^{-d} \sum_{x \in \Lambda(1/h)} \sigma_x, \qquad \forall \sigma \in \Omega(\Lambda(1/h)) \tag{2.4}$$

The distribution of $X_{\Lambda(1/h)}$ under $\mu_{\Lambda(1/h), \eta, T, s}$ is denoted by $P_{\Lambda(1/h), \eta, T, s}$. In particular, $P_{\Lambda(1/h), \eta, T, 0}$ is the distribution of $X_{\Lambda(1/h)}$ with respect to the Gibbs measure without external field.

Definition. Let μ_h be probability measures on R, $h>0$. Let α_h be positive constants such that $\alpha_h \to \infty$ as $h \searrow 0$. We say that $\{\mu_h\}$ satisfies a large-deviation principle (LDP) with constants (α_h) if there exists a lower semicontinuous (lsc) function $I: R \to [0, \infty]$ such that for all open sets $G \subset R$ and closed sets $F \subset R$

$$\limsup_{h \searrow 0} \frac{1}{\alpha_h} \log \mu_h(F) \leqslant -\inf_{x \in F} I(x) \tag{2.5}$$

$$\liminf_{h \searrow 0} \frac{1}{\alpha_h} \log \mu_h(G) \geqslant -\inf_{x \in G} I(x) \tag{2.6}$$

The function I is called the rate function. The LDP for $h \searrow 0$ along a subsequence is defined similarly.

Theorem 2.1. Suppose that, for some $a_0 \geqslant 0$, $\{P_{\Lambda(1/h), \eta, T, a_0 h}\}$ satisfies an LDP with constants $((1/h)^{d-1})$ and rate function I_{a_0}, then for all $a \geqslant 0$, $\{P_{\Lambda(1/h), \eta, T, ah}\}$ satisfies an LDP with the same constants $((1/h)^{d-1})$ and rate function

$$I_a(x)=I_{a_0}(x)-\frac{\beta(a-a_0)}{2}x+I_{a_0}^*\left(\frac{\beta(a-a_0)}{2}\right), \qquad \forall x \in R \tag{2.7}$$

where $I_{a_0}^*$ is the convex conjugate of I_{a_0} defined by

$$I_{a_0}^*(t):=\sup\{tx-I_{a_0}(x): x \in R\}, \qquad \forall t \in R \tag{2.8}$$

The second term on the right side of (2.7) comes from the difference of the external fields, while the third comes from the difference of the normalizing constants. Note that the LDP's in this theorem have the rate $((1/h)^{d-1})$, which is appropriate for phase coexistence regions. This theorem says, in particular, that if we have an LDP without external field, then we have a corresponding LDP with external field. The only result we know giving an LDP without external field is that of Schonmann and Shlosman[13] for the 2D Ising model, which extends results of Ioffe.[4,5] However, in higher dimension, even without such an LDP, we can apply this theorem along subsequences to obtain a critical value of B.

Since the distribution of $X_{\Lambda(B/h)}$ under the Gibbs measure $\mu_{\Lambda(B/h),\eta,T,h}$ is $P_{\Lambda(1/h'),\eta,T,Bh'}$, with $h'=h/B$, we have the following corollary:

Corollary 2.2. If $\{P_{\Lambda(1/h),\eta,T,0}\}$, with no external field ($a_0=0$), satisfies an LDP with constants $((1/h)^{d-1})$ and rate function $I_0(x)$, then $\{P_{\Lambda(B/h),\eta,T,h}\}$, with the external field h, satisfies an LDP with constants $((B/h)^{d-1})$ and rate function $I_B(x)$, and vice versa. Moreover, the rate functions are related by

$$I_B(x)=I_0(x)-\frac{\beta B}{2}x+I_0^*\left(\frac{\beta B}{2}\right), \qquad \forall x\in R \tag{2.9}$$

3. APPLICATIONS TO THE ISING MODEL

The discussion in this section can be extended to all finite-range ferromagnetic models. Let $E=\{-1,1\}$ and let the energy $H_{\Lambda,\eta}$ in (2.1) be

$$H_{\Lambda,\eta}(\sigma):=-\sum_{\{x,y\}\subset\Lambda,\,|x-y|=1}\sigma_x\sigma_y-\sum_{x\in\Lambda,\,y\in\Lambda^c,\,|x-y|=1}\sigma_x\eta_y \tag{3.1}$$

When $\eta\equiv-1$ or $+1$, we replace η by $-$ or $+$. It is well-known that $\mu_{\Lambda(1/h),-,T,s}$ (resp. $\mu_{\Lambda(1/h),+,T,s}$) converges weakly as $h\searrow 0$ to a probability measure denoted by $\mu_{-,T,s}$ (resp $\mu_{+,T,s}$). There is a critical temperature $T_c>0$ such that for all $T<T_c$, the $(-)$ phase $\mu_{-,T,0}$ and the $(+)$ phase $\mu_{+,T,0}$ are different, in which case a phase transition occurs. The spontaneous magnetization $m_T^*:=E_{+,T,0}[\sigma_0]$ is positive for all T subcritical. It is also known that without external field,

$$X_{\Lambda(1/h)}\to -m_T \tag{3.2}$$

a.s. with respect to $\mu_{-,T,0}$ as $h\searrow 0$. The convergence is not exponential at the rate $(1/h)^d$, as one might expect, but may be exponential at the rate $(1/h)^{d-1}$, the order of the volume of the boundary of $\Lambda(1/h)$. This was first proved by Schonmann[10] for low temperatures. An interesting related problem involves the competing influences of the Gibbs measure in $\Lambda(B/h)$ with external field h and negative boundary condition as $h\searrow 0$ for various B. Martirosyan[6] showed that when T is small, there is a $(+)$ phase in the cube $\Lambda(B/h)$ as $h\searrow 0$ for all B greater than some $B_2(T)$. Schonmann,[11] among others, re-proved Martirosyan's result and extended it by showing that only the $(-)$ phase appears in $\Lambda(B/h)$ as $h \searrow 0$ when T is small and B is less than some $B_1(T)$. As noted by Schonmann,[11] it is unknown whether $B_1(T) = B_2(T)$ in general. But when $d=2$, Schonmann and Shlosman[13] completely solved this problem by proving the existence of the critical value B_0 for all subcritical T and giving it explicitly. In this section, we will look at the same problem for all $d \geqslant 2$ and subcritical T. However, we consider average spins rather than $(+)$ or $(-)$ phases because we have large deviation techniques only in this case. We will prove that there exists a critical value B_0 such that $X_{\Lambda(B/h)}$ converges to $-m_T^*$ under $\mu_{\Lambda(B/h),\,-,\,T,\,h}$ as $h \searrow 0$ only when $B < B_0$ (Theorem 3.2).

When $d=2$, we will see that our B_0 is the same as that of Schonmann and Shlosman,[13] using their LDP. Their methods are more sophisticated and establish the *squeezed Wulff shape*. Our main contribution is to provide some insight for the case $d > 2$.

To state our theorems, we need a preparatory lemma. Define, for all $t \in R$,

$$\bar{\Phi}(t) = \limsup_{h \searrow 0} h^{d-1} \log E_{\Lambda(1/h),\,-,\,T,\,0}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})] \tag{3.3}$$

Note that $\bar{\Phi}(t)$ is a finite, continuous convex function in R. Define

$$B_0 = B_0(T) := 2T \sup\{t \colon \bar{\Phi}(t) = -m_T^* t\} \tag{3.4}$$

Lemma 3.1. Let $d \geqslant 2$ and let T be subcritical. Then:

(a) For all $t \leqslant 0$, $\bar{\Phi}(t) = -m_T^* t$ and for some $t > 0$, $\bar{\Phi}(t) > -m_T^* t$, so that $0 \leqslant B_0 < \infty$.

(b) If there exists a $B > 0$ such that

$$\lim_{h \searrow 0} E_{\Lambda(B/h),\,-,\,T,\,h}[\sigma_0] = -m_T^* \tag{3.5}$$

then $B_0 > 0$. In particular,[11] B_0 is positive for all sufficiently small $T > 0$.

(c) Define $\bar{I} := \bar{\Phi}^*$, the convex conjugate of $\bar{\Phi}$, and

$$\bar{I}_B(x) := \bar{I}(x) - \frac{\beta B}{2} x + \bar{\Phi}\left(\frac{\beta B}{2}\right), \qquad \forall x \in R \tag{3.6}$$

Then for all $B > 0$, $\bar{I}_B$ is a nonnegative lsc convex function, and the set $\mathscr{F}_B := \{x \colon \bar{I}(x) = 0\}$ is nonempty.

(d) For all $0 < B < B_0$, $\mathscr{F}_B = \{-m_T^*\}$. For all $B > B_0$, $\mathscr{F}_B \subset (-m_T^*, \infty)$. $\mathscr{F}_B$ is a singleton except for countably many $B > 0$.

By Lemma 3.1(c), we can define $m(B) := \inf_{x \in \mathscr{F}_B} x$. It turns out that $m(B)$ will be the weak limit of $X_{A(B/h)}$ under $\mu_{A(B/h), -, T, h}$ as $h \searrow 0$ when $0 < B < B_0$, and as $h \searrow 0$ along some subsequence, when $B > B_0$.

Theorem 3.2. Let $d \geqslant 2$ and let T be subcritical. Define the constants B_0 and $m(B)$ as above. Then:

(a) If $0 < B < B_0$, then for all $\varepsilon > 0$,

$$\limsup_{h \searrow 0} \left(\frac{h}{B}\right)^{d-1} \log \mu_{A(B/h), -, T, h}(|X_{A(B/h)} - (-m_T^*)|) \leqslant -\frac{\beta(B_0 - B)\,\varepsilon}{2} \tag{3.7}$$

(b) For all $B > B_0$, if $\mathscr{F}_B$ is a singleton [true for almost all $B > B_0$ by Lemma 3.1(d)], then $m(B) > -m_T^*$ and there exists a subsequence $h_n \searrow 0$ such that for all $\varepsilon > 0$.

$$\limsup_{n \to \infty} \left(\frac{h_n}{B}\right)^{d-1} \log \mu_{A(B/h_n), -, T, h_n}(|X_{A(B/h_n)} - m(B)| \geqslant \varepsilon) \leqslant -c_\varepsilon \tag{3.8}$$

where $c_\varepsilon = \bar{I}_B(m(B) - \varepsilon) \wedge \bar{I}_B(m(B) + \varepsilon) > 0$.

In Theorem 3.2, we see that for $B < B_0$, the asymptotics of the average spins is like that for the $(-)$ phase, whereas for $B > B_0$, the asymptotics of the average spins is different. Hence B_0 is a critical point of B.

In the case $d = 2$, we will see in Theorem 3.3 that $\mathscr{F}_B$ in Lemma 3.1(d) is a singleton for all $B > B_0$ and that the convergence in Theorem 3.2(b) holds along the whole sequence $h \searrow 0$. It seems to us there is no reason to believe this fails for $d > 2$, although our method does not yield the result.

Let us now consider $d = 2$. Then Theorem 1 of Schonmann and Shlosman[13] says that for all subcritical T, $\{P_{A(1/h), -, T, 0}\}$ satisfies an LDP with constants $(1/h)$ and rate function given by

$$I(x) = \beta \begin{cases} \omega_T \left(\dfrac{m_T^* + x}{2m_T^*}\right)^{1/2} & \text{if} \quad -m_T^* \leqslant x \leqslant \alpha_T \\ 4\bar{\tau}_T - \sqrt{\delta_T}\sqrt{m_T^* - x} & \text{if} \quad \alpha_T \leqslant x \leqslant m_T^* \\ \infty & \text{otherwise} \end{cases} \tag{3.9}$$

where $\delta_T = (16\bar{\tau}_T^2 - \omega_T^2)/(2m_T^*)$, $\alpha_T = 2m_T^* l_T^{-2} - m_T^*$, and ω_T, $\bar{\tau}_T$, and l_T are positive constants, which are explicitly given in terms of the *Wulff functional*, *Wulff shape* and *squeezed Wulff shape*. Putting this theorem together with our Corollary 2.2 and Lemma 3.1, we have the following result.

Theorem 3.3. Consider the two-dimensional Ising model below the critical temperature. Let I be given by (3.9). Then.

(a) For all $B>0$, the distribution $\{P_{\Lambda(B/h),\,-,\,T,\,h}\}$ of the average spin $X_{\Lambda(B/h)}$ under $\mu_{\Lambda(B/h),\,-,\,T,\,h}$ satisfies an LDP with constants (B/h) and rate function I_B given by

$$I_B(x)=I(x)-\frac{\beta B}{2}x+I^*\left(\frac{\beta B}{2}\right),\qquad \forall x\in R \tag{3.10}$$

(b) The B_0 defined in (3.4) equals $(4\bar{\tau}_T+\omega_T)/(2m_T^*)$.

(c) For all $B\neq B_0$, $I_B(x)=0$ has a unique solution given by

$$m(B)=\begin{cases}-m_T^* & \text{if } 0<B<B_0\\ m_T^*-\delta_T/B^2 & \text{if } B>B_0\end{cases} \tag{3.11}$$

So, in particular, the LDP in (a) implies that $X_{\Lambda(B/h)}$ converges exponentially to $m(B)$ under $\mu_{\Lambda(B/h),\,-,\,T,\,h}$ at the rate (B/h) as $h\searrow 0$.

(d) $I_{B_0}(x)=0$ has exactly two solutions m_T^* and $m(B_0):=m_T^*-\delta_T/B_0^2$. So the LDP in (a) implies that only m_T^* or $m(B_0)$ may possibly be a weak limit of $X_{\Lambda(B_0/h)}$ under $\mu_{\Lambda(B_0/h),\,-,\,T,\,h}$ as $h\searrow 0$, and the weak convergence, *if it exists*, is not exponential at the rate $(1/h)$.

Remark 3.1. (a) The critical value B_0 and the limits $m(B)$ are the same as those in Schonmann and Shlosman.[13] Our large-deviation result in (a) refines their Theorem 2a(1) and 2b(1).

(b) Note that, if the critical value B_0 in Theorem 3.2 is positive, then $X_{\Lambda(1/h)}$ converges to $-m_T^*$ under $\mu_{\Lambda(1/h),\,-,\,T,\,0}$ exponentially at the rate $(1/h)^{d-1}$ as $h\searrow 0$. We believe the converse holds, too. But we do not have a proof. A problem which seems out of reach at present is to prove that, when $d>2$ and $B>B_0$, there also appears a droplet of $(+)$ phase in the box $\Lambda(1/h)$ with negative boundary condition as Schonmann and Shlosman[13] describe for $d=2$ in their Theorem 2.

4. THE PROOFS

We will need an analogue for large deviations of Prohorov's Theorem for weak convergence. This result about compactness in large deviation theory was proved by O'Brien and Vervaat[8] and extended by O'Brien.[7] We state the result for the bounded family of random variables $X_{\Lambda(1/h)}$.

Proposition 4.1. Consider

$$\{P_{\Lambda(1/h),\,\eta,\,T,\,ah}(dx)=\mu_{\Lambda(1/h),\,\eta,\,T,\,ah}(X_{\Lambda(1/h)}\in dx)\},\qquad a\geqslant 0$$

Then for any subsequence of (h), there exists a subsubsequence $h_n \searrow 0$ such that $\{P_{\Lambda(1/h_n), \eta, T, ah_n}\}$ satisfies an LDP with the constants $(1/h_n)^{d-1}$ and some rate function $\mathscr{I}$. Moreover, by Varadhan's Integral Theorem,[1] the pressure function $\Psi(t)$ for this subsubsequence exists and

$$\Psi(t) := \lim_{n \to \infty} h_n^{d-1} \log E_{\Lambda(1/h_n), \eta, T, ah_n}[\exp(t(1/h_n)^{d-1} X_{\Lambda(1/h_n)})] \quad (4.1)$$

$$= \sup\{tx - \mathscr{I}(x): x \in R\} = \mathscr{I}^*(t), \qquad \forall t \in R \quad (4.2)$$

In particular,

$$\mathscr{I}(x) \geqslant \operatorname{conv}(\mathscr{I})(x) = \mathscr{I}^{**}(x) = \Psi^*(x), \qquad \forall x \in R \quad (4.3)$$

where $\operatorname{conv}(\mathscr{I})$, called the convex hull of $\mathscr{I}$, is the greatest lsc convex function not exceeding $\mathscr{I}$.[9]

Proof of Theorem 2.1. Let $a \geqslant 0$. Let $x \in R$ and $\varepsilon > 0$. Consider

$$\frac{\mu_{\Lambda(1/h), \eta, T, ah}(X_{\Lambda(1/h)} \in (x-\varepsilon, x+\varepsilon))}{\mu_{\Lambda(1/h), \eta, T, a_0 h}(X_\Lambda(1/h) \in (x-\varepsilon, x+\varepsilon))}$$

$$= \frac{\sum_{X_{\Lambda(1/h)}(\sigma) \in (x-\varepsilon, x+\varepsilon)} \exp(-\beta H_{\Lambda(1/h), \eta, ah}(\sigma))}{\sum_{X_{\Lambda(1/h)}(\sigma) \in (x-\varepsilon, x+\varepsilon)} \exp(-\beta H_{\Lambda(1/h), \eta, a_0 h}(\sigma))} \cdot \frac{Z_{\Lambda(1/h), \eta, T, a_0 h}}{Z_{\Lambda(1/h), \eta, T, ah}}$$

$$=: \Delta_1 \cdot \Delta_2 \quad (4.4)$$

Note that

$$E_{\Lambda(1/h), \eta, T, a_0 h}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})]$$

$$= \frac{\sum_{\sigma \in \Omega(\Lambda(1/h))} \exp(-\beta H_{\Lambda(1/h), \eta, a_0 h}(\sigma) + th \sum_{x \in \Lambda(1/h)} \sigma_x)}{Z_{\Lambda(1/h), \eta, T, a_0 h}}$$

$$= \frac{Z_{\Lambda(1/h), \eta, T, (a_0 + 2tT)h}}{Z_{\Lambda(1/h), \eta, T, a_0 h}}$$

$$[= \Delta_2^{-1} \qquad \text{if} \quad t = (a - a_0)/(2T)] \quad (4.5)$$

Note that $X_{\Lambda(1/h)}$ is bounded for all $h > 0$. So by Varadhan's Integral Theorem[1] one has

$$\Phi_{a_0}(t) := \lim_{h \searrow 0} h^{d-1} \log E_{\Lambda(1/h), \eta, T, a_0 h}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})]$$

$$= \sup\{tx - I_{a_0}(x): x \in R\} = I_{a_0}^*(t), \qquad \forall t \in R \quad (4.6)$$

So, letting $t=\beta(a-a_0)/2$ in (4.5) and (4.6), we see that

$$\begin{aligned}
&\lim_{h\searrow 0} h^{d-1}\log \Delta_2\\
&\quad = -\lim_{h\searrow 0} h^{d-1}\log E_{\Lambda(1/h),\eta,T,a_0h}\left\{\exp\left[\frac{\beta(a-a_0)}{2}\left(\frac{1}{h}\right)^{d-1} X_{\Lambda(1/h)}\right]\right\}\\
&\quad = -\Phi_{a_0}\left(\frac{\beta(a-a_0)}{2}\right) = -I^*_{a_0}\left(\frac{\beta(a-a_0)}{2}\right)
\end{aligned} \tag{4.7}$$

Note that

$$\begin{aligned}
&\log \Delta_1\\
&\quad = \int_{a_0h}^{ah}\frac{d}{ds}\left\{\log \sum_{X_{\Lambda(1/h)}(\sigma)\in(x-\varepsilon,x+\varepsilon)} \exp[-\beta H_{\Lambda(1/h),\eta,s}(\sigma)]\right\}ds\\
&\quad = \int_{a_0h}^{ah}\frac{\sum_{X_{\Lambda(1/h)}(\sigma)\in(x-\varepsilon,x+\varepsilon)}\frac{1}{2}\beta\left(\sum_{x\in\Lambda(1/h)}\sigma_x\right)\exp(-\beta H_{\Lambda(1/h),\eta,s}(\sigma))}{\sum_{X_{\Lambda(1/h)}(\sigma)\in(x-\varepsilon,x+\varepsilon)}\exp(-\beta H_{\Lambda(1/h),\eta,s}(\sigma))}ds\\
&\quad = \int_{a_0h}^{ah}\frac{\sum_{X_{\Lambda(1/h)}(\sigma)\in(x-\varepsilon,x+\varepsilon)}\frac{1}{2}\beta(1/h)^d X_{\Lambda(1/h)}(\sigma)\exp(-\beta H_{\Lambda(1/h),\eta,s}(\sigma))}{\sum_{X_{\Lambda(1/h)}(\sigma)\in(x-\varepsilon,x+\varepsilon)}\exp(-\beta H_{\Lambda(1/h),\eta,s}(\sigma))}ds
\end{aligned} \tag{4.8}$$

So by the mean value theorem, $\log \Delta_1=(a-a_0)\,hC_{h,\varepsilon}$, where the constant $C_{h,\varepsilon}$ satisfies

$$\frac{\beta}{2}\left(\frac{1}{h}\right)^d(x-\varepsilon)\leqslant C_{h,\varepsilon}\leqslant\frac{\beta}{2}\left(\frac{1}{h}\right)^d(x+\varepsilon)$$

Therefore,

$$\lim_{\varepsilon\searrow 0}\lim_{h\searrow 0} h^{d-1}\log \Delta_1=\frac{\beta(a-a_0)}{2}x \tag{4.9}$$

Using (4.9), (4.7), and (4.4), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\lim_{h\searrow 0} h^{d-1}\log \mu_{\Lambda(1/h),\eta,T,ah}(X_{\Lambda(1/h)}\in(x-\varepsilon,x+\varepsilon))\\
&\quad = \lim_{\varepsilon\searrow 0}\lim_{h\searrow 0} h^{d-1}\log \mu_{\Lambda(1/h),\eta,T,a_0h}(X_{\Lambda(1/h)}\in(x-\varepsilon,x+\varepsilon))\\
&\qquad +\frac{\beta(a-a_0)}{2}x - I^*_{a_0}\left(\frac{\beta(a-a_0)}{2}\right)
\end{aligned} \tag{4.10}$$

if either of the two limits exists. Since $\{\mu_{A(1/h), \eta, T, a_0 h}(X_{A(1/h)} \in dx)\}$ satisfies an LDP with the constants $((1/h)^{d-1})$ and rate function I_{a_0}, by a standart expression for the rate function, (ref. 2, p. 35), we have

$$I_{a_0}(x) = -\lim_{\varepsilon \searrow 0} \lim_{h \searrow 0} h^{d-1} \log \mu_{A(1/h), \eta, T, a_0 h}(X_{A(1/h)} \in (x-\varepsilon, x+\varepsilon)), \qquad \forall x \in R \tag{4.11}$$

Actually, the limit in (4.11) with respect to $h \searrow 0$ exists only for almost all $\varepsilon > 0$. But this limit expression will not cause confusion because of the monotonicity (of the limits with respect to $h \searrow 0$) in $\varepsilon > 0$. So the limit in the left side of (4.10) exists, too. Define

$$I_a(x) = -\lim_{\varepsilon \searrow 0} \lim_{h \searrow 0} h^{d-1} \log \mu_{A(1/h), \eta, T, a_h}(X_{A(1/h)} \in (x-\varepsilon, x+\varepsilon)), \qquad \forall x \in R \tag{4.12}$$

Then (2.7) follows from (4.10). It now remains to prove that I_a is the correct large-deviation rate function of $\{\mu_{A(1/h), \eta, T, ah}(X_{A(1/h)} \in dx)\}$ with the constants $((1/h)^{d-1})$. First of all, since I_{a_0} is lsc, I_a is also lsc. The lower bound (2.6) is an immediate result of the definition of I_a. The upper bound (2.5) for all compact sets follows from (4.12) and a standard argument. So the upper bound (2.5) holds for all closed sets, because $X_{A(1/h)}$ is bounded. ▌

Remark 4.1. The above proof indicates that, if the condition of Theorem 2.1 holds only along a subsequence $h_n \searrow 0$, then the conclusion holds along the same subsequence.

Proof of Lemma 3.1. (a) Using (4.5) with $a_0 = 0$ and $\eta \equiv -1$, we obtain

$$\begin{aligned}
&\log E_{A(1/h), -, T, 0}\left\{\exp\left[t\left(\frac{1}{h}\right)^{d-1} X_{A(1/h)}\right]\right\} \\
&= \log \frac{Z_{A(1/h), -, T, 2tTh}}{Z_{A(1/h), -, T, 0}} \\
&= \int_0^{2tTh} \frac{d}{ds}(\log Z_{A(1/h), -, T, s})\, ds \\
&= \frac{\beta}{2}\left(\frac{1}{h}\right)^d \int_0^{2tTh} E_{A(1/h), -, T, s}[X_{A(1/h)}]\, ds \\
&= \left(\frac{1}{h}\right)^{d-1} \int_0^t E_{A(1/h), -, T, 2Thu}[X_{A(1/h)}]\, du
\end{aligned} \tag{4.13}$$

By the FKG inequality, for all $s_0 < s < 0$.

$$E_{A(1/h),\,-,\,T,\,s_0}[X_{A(1/h)}] \leqslant E_{A(1/h),\,-,\,T,\,s}[X_{A(1/h)}]$$
$$\leqslant E_{A(1/h),\,-,\,T,\,0}[X_{A(1/h)}] \leqslant E_{-,\,T,\,0}[X_{A(1/h)}]$$

The last of these quantities obviously converges to $-m_T^*$ as $h \searrow 0$. Also the first converges to $-m_T^*$ as $h \searrow 0$ and then $s_0 \nearrow 0$.[3] So if $t \leqslant 0$, then (4.13) implies that the limit (not only the superior limit) in (3.3) exists and equals $-m_T^* t$, and so $\bar{\Phi}(t) = -m_T^* t$. Next, by changing the negative boundary condition to positive and using Jensen's inequality, one has

$$\log E_{A(1/h),\,-,\,T,\,0}\{\exp[t(1/h)^{d-1} X_{A(1/h)}]\}$$
$$\geqslant \log(\exp(-4\beta d(1/h)^{d-1}) \cdot E_{A(1/h),\,+,\,T,\,0}\{\exp[t(1/h)^{d-1} X_{A(1/h)}]\})$$
$$\geqslant -4\beta d(1/h)^{d-1} + t(1/h)^{d-1} E_{A(1/h),\,+,\,T,\,0}[X_{A(1/h)}]$$

So $\bar{\Phi}(t) \geqslant -4\beta d + m_T^* t$ for all $t > 0$. This proves $B_0 < \infty$.

(b) By (4.13) and the FKG inequality, for all $t > 0$,

$$\bar{\Phi}(t) \leqslant t \limsup_{h \searrow 0} E_{A(1/h),\,-,\,T,\,2Tt/h}[X_{A(1/h)}]$$
$$= t \limsup_{h \searrow 0} E_{A(2Tt/h),\,-,\,T,\,h}[X_{A(2Tt/h)}] \tag{4.14}$$

For each $x \in A(2Tt/h)$, the cube $A(4Tt/h)$ centered at x contains $A(2Tt/h)$. So by the FKG inequality and (4.14),

$$\bar{\Phi}(t) \leqslant t \limsup_{h \searrow 0} E_{A(4Tt/h),\,-,\,T,\,h}[\sigma_0]$$

Therefore, if the condition (3.5) holds, then $\bar{\Phi}(t) \leqslant -m_T^* t$ at $t = B/(4T)$. This implies $\bar{\Phi}(t) = -m_T^* t$ for all small $t > 0$, since $\bar{\Phi}(t)$ is convex. Finally, Theorem 2 of Schonmann[11] says that the condition (3.5) holds for all small $T > 0$.

(c) Since for all $x \in R$

$$\bar{I}_B(x) = \sup\{xt - \bar{\Phi}(t) \colon t \in R\} - \left(\frac{\beta B}{2} x - \bar{\Phi}\left(\frac{\beta B}{2}\right)\right)$$

$\bar{I}_B$ is nonnegative. $\bar{I}_B$ is convex and lsc because $\bar{I}$ is. It remains to prove $\mathscr{F}_B \neq \varnothing$. Let $B > 0$. Then we can choose a subsequence $h_n \searrow 0$ as $n \to \infty$ such that

$$\bar{\Phi}\left(\frac{\beta B}{2}\right) = \lim_{n \to \infty} h_n^{d-1} \log E_{A(1/h_n),\,-,\,T,\,0}\left\{\exp\left[\frac{\beta B}{2}\left(\frac{1}{h_n}\right)^{d-1} X_{A(1/h_n)}\right]\right\} \tag{4.15}$$

By Proposition 4.1, we may assume that $\{\mu_{A(1/h_n), -, T, 0}(X_{A(1/h_n)} \in dx)\}$ satisfies an LDP with the constants $((1/h_n)^{d-1})$ and some rate function $\mathscr{I}$, because otherwise one needs only to choose a subsubsequence, which still satisfies (4.15). Then by Theorem 2.1 and Remark 4.1, $\{\mu_{A(1/h_n), -, T, Bh_n}(X_{A(1/h_n)} \in dx)\}$ satisfies an LDP with the same constants and rate function given by

$$\mathscr{I}_B(x) = \mathscr{I}(x) - \frac{\beta B}{2}x + \mathscr{I}^*\left(\frac{\beta B}{2}\right)$$

$$= \mathscr{I}(x) - \frac{\beta B}{2}x + \bar{\Phi}\left(\frac{\beta B}{2}\right), \qquad \forall x \in R \tag{4.16}$$

where the last step follows from (4.2) with $a = 0$ at $t = \beta B/2$ and (4.15). Note that the corresponding pressure function $\Psi(t)$ along this subsequence $h_n \searrow 0$ [given in (4.1)] is obviously not greater than $\bar{\Phi}$. So by (4.3), $\mathscr{I}(x) \geqslant \Psi^*(x) \geqslant \bar{\Phi}^*(x) = \bar{I}(x)$. We obtain $\mathscr{I}_B \geqslant \bar{I}_B$ from (4.16). Since $\mathscr{I}_B$ is a rate function, $\mathscr{I}_B(x) = 0$ has at least one solution, and so does $\bar{I}_B(x) = 0$.

(d) Let $\alpha = \bar{I}'_+(-m_T^*)$. Note that $\bar{\Phi}'_-(\beta B_0/2) = -m_T^*$ and $\bar{\Phi}'_-(t) > -m_T^*$ if $t > \beta B_0/2$. Then by Theorem 23.5 of Rockafellar,[9] $-m_T^*$ is a subgradient of $\bar{\Phi}$ at α and $\beta B_0/2$ is a subgradient of $\bar{I}$ at $-m_T^*$. Hence $\alpha \leqslant \beta B_0/2$ and $\beta B_0/2 \leqslant \alpha$. So $\bar{I}'_+(-m_T^*) = \beta B_0/2$. Note that the definition of B_0 in (3.4) implies that

$$\bar{\Phi}\left(\frac{\beta B}{2}\right) = -\frac{\beta B}{2} m_T^*$$

for all $B < B_0$, so by the definition of $\bar{I}_B$ in (3.6), for all $0 < B < B_0$,

$$\bar{I}_B(x) = \bar{I}(x) - \frac{\beta B}{2}(x + m_T^*) \geqslant \frac{\beta(B_0 - B)}{2}(x + m_T^*) \tag{4.17}$$

This and (c) prove that $\mathscr{F}_B = \{-m_T^*\}$, when $B < B_0$. Now assume that $B > B_0$. If $\bar{I}_B(-m_T^*) = 0$, then the definition of $\bar{I}_B$ implies that $\bar{\Phi}(\beta B/2) = -m_T^* \beta B/2$. This contradicts the definition of B_0. So $\mathscr{F}_B \subset (-m_T^*, \infty)$. Finally, by (a)

$$\bar{I}_B(x) = \bar{I}(x) - \left[\frac{\beta B}{2}x - \bar{\Phi}\left(\frac{\beta B}{2}\right)\right] \geqslant 0$$

i.e., the line $(\beta B/2)x - \bar{\Phi}(\beta B/2)$ lies below the graph of $\bar{I}$. Recall that $\mathscr{F}_B$ is the set of x where the line and the graph intersect. If $\mathscr{F}_B$ has more than one point, then $\bar{I}$ has a linear segment with slope $\beta B/2$. Since $\bar{I}$ has at most countably many such segments, $\mathscr{F}_B$ must be a singleton except for countably many $B>0$. ∎

Proof of Theorem 3.2. (a) For all $\varepsilon>0$, choose a subsequence (h_n) such that

$$\limsup_{h\searrow 0}\left(\frac{h}{B}\right)^{d-1}\log\mu_{\Lambda(B/h),\,-,\,T,\,h}(|X_{\Lambda(B/h)}-(-m_T^*)|\geqslant\varepsilon)$$
$$=\lim_{n\to\infty}\left(\frac{h_n}{B}\right)^{d-1}\log\mu_{\Lambda(B/h_n),\,-,\,T,\,h_n}(|X_{\Lambda(B/h_n)}-(-m_T^*)|\geqslant\varepsilon) \qquad (4.18)$$

Let $h_n'=h_n/B$. Then by Proposition 4.1, one may assume that

$$\{\mu_{\Lambda(1/h_n'),\,-,\,T,\,0}(X_{\Lambda(1/h_n')}\in dx)\}$$

satisfies an LDP with constants $((1/h_n')^{d-1})$ and some rate function $\mathscr{I}$. Then by Theorem 2.1 and Remark 4.1,

$$\{\mu_{\Lambda(B/h_n),\,-,\,T,\,h_n}(X_{\Lambda(B/h_n)}\in dx)\}=\{\mu_{\Lambda(1/h_n'),\,-,\,T,\,Bh_n'}(X_{\Lambda(1/h_n')}\in dx)\}$$

satisfies an LDP with the constants $(B/h_n)^{d-1}=(1/h_n')^{d-1}$ and the rate function $\mathscr{I}_B$ given by

$$\mathscr{I}_B(x)=\mathscr{I}(x)-\frac{\beta B}{2}x+\mathscr{I}^*\left(\frac{\beta B}{2}\right) \qquad (4.19)$$

Since $\bar{\Phi}^*(x)=\infty$ for all $x<-m_T^*$, the large-deviation upper bound[14] implies that $\mu_{\Lambda(1/h),\,-,\,T,\,0}(X_{\Lambda(1/h)}\leqslant -m_T^*-\varepsilon)$ converges to 0 as $h\searrow 0$. The FKG inequality implies that

$$\mu_{\Lambda(1/h),\,-,\,T,\,0}(X_{\Lambda(1/h)}\geqslant -m_T^*+\varepsilon)\leqslant\mu_{-,\,T,\,0}(X_{\Lambda(1/h)}\geqslant -m_T^*+\varepsilon)$$

which converges to 0 as $h\searrow 0$, as pointed out in (3.2). So $X_{\Lambda(1/h)}$ converges to $-m_T^*$ in $\mu_{\Lambda(1/h),\,-,\,T,\,0}$ as $h\searrow 0$. Hence $\mathscr{I}(-m_T^*)=0$. Letting $x=-m_T^*$ in (4.19), we obtain

$$\frac{\beta B}{2}m_T^*+\mathscr{I}^*\left(\frac{\beta B}{2}\right)=\mathscr{I}_B(-m_T^*)\geqslant 0$$

As in the proof of Lemma 3.1(c), (4.3) implies that $\mathscr{I} \geqslant \bar{I}$. So

$$\mathscr{I}_B(x) = \mathscr{I}(x) - \frac{\beta B}{2}(x + m_T^*) + \frac{\beta B}{2} m_T^* + \mathscr{I}^*\left(\frac{\beta B}{2}\right)$$

$$\geqslant \bar{I}(x) - \frac{\beta B}{2}(x + m_T^*) \geqslant \frac{\beta(B_0 - B)}{2}(x + m_T^*) \qquad (4.20)$$

where the last step follows from $\bar{I}'_+(-m_T^*) = \beta B_0/2$, seen in the proof of Lemma 3.1(d). So the large deviation upper bound (2.5) for $\{\mu_{A(B/h_n), -, T, h_n}(X_{A(B/h_n)} \in dx)\}$ implies that, when $B < B_0$,

$$\limsup_{n\to\infty} \left(\frac{h_n}{B}\right)^{d-1} \log \mu_{A(B/h_n), -, T, h_n}(|X_{A(1/h_n)} - (-m_T^*)| \geqslant \varepsilon)$$

$$\leqslant - \inf_{|x - (-m_T^*)| \geqslant \varepsilon} \mathscr{I}_B(x)$$

$$\leqslant - \inf_{x \geqslant -m_T^* + \varepsilon} \left[\bar{I}(x) - \frac{\beta B}{2}(x + m_T^*)\right] \leqslant - \frac{\beta(B_0 - B)\varepsilon}{2}$$

This and (4.18) prove (3.7).

(b) Let $B > B_0$ be such that $\mathscr{F}_B$ is a singleton. By Lemma 3.1(d), $m(B) > -m_T^*$ is the unique point where $\bar{I}_B$ is 0. By the convexity, $\bar{I}_B(x)$ must be nonincreasing in $(-\infty, m(B)]$ and nondecreasing in $[m(B), \infty)$. For any such $B > B_0$, repeating the argument in the proof of Lemma 3.1(c), one concludes that there exists a subsequence $h_n \searrow 0$ as $n \to \infty$ such that $\{\mu_{A(1/h_n), -, T, Bh_n}(X_{A(1/h_n)} \in dx)\}$ satisfies an LDP with the constants $((1/h_n)^{d-1})$ and rate function $\mathscr{I}_B \geqslant \bar{I}_B$. So the large-deviation upper bound (2.5) implies

$$\limsup_{n\to\infty} h_n^{d-1} \log \mu_{A(1/h_n), -, T, Bh_n}(|X_{A(1/h_n)} - m(B)| \geqslant \varepsilon)$$

$$\leqslant - \inf_{|x - m(B)| \geqslant \varepsilon} \mathscr{I}_B(x) \leqslant - \inf_{|x - m(B)| \geqslant \varepsilon} \bar{I}(x) = -c_\varepsilon < 0 \qquad (4.21)$$

for all $\varepsilon > 0$. This proves (3.8) along the subsequence (Bh_n). ∎

Remark 4.2. The main reason that the above proof of (a) fails in (b) is that the $\mathscr{I}^*(\beta B/2)$ in (4.19) is the limit of (3.3) along a subsequence (h_n) and hence is less than or equal to $\bar{\Phi}(\beta B/2)$ in general. When $B < B_0$, one can go further, to (4.20). But (4.20) is useless when $B > B_0$. If one knows that the limit in (3.3) exists, then (3.8) holds along the whole sequence $h \searrow 0$.

Proof of Theorem 3.3. Part (a) follows from Corollary 2.2. (b) First of all, by Varadhan's Integral Theorem, $\bar{\Phi} = I^*$. The convex conjugate I^* of I is $(\text{conv}(I))^*$. Notice that the rate function I is concave in $[-m_T^*, \alpha_T]$ and convex in $[\alpha_T, m_T^*]$, so that $\text{conv}(I)(x)$ must be linear for x from $-m_T^*$ to some $x_0 \geqslant \alpha_T$ and equal to $I(x)$ afterwards. This x_0 may be obtained from the condition that the line connecting the points $(-m_T^*, 0)$ and $(x_0, I(x_0))$ is the line tangent to I at $(x_0, I(x_0))$. By solving

$$\frac{I(x_0)}{x_0 + m_T^*} = I'(x_0), \qquad x_0 \geqslant \alpha_T$$

we get

$$x_0 = [1 - 2(4\bar{\tau}_T - \omega_T)/(4\bar{\tau}_T + \omega_T)]\, m_T^*$$

$$I'(x_0) = \frac{1}{2T} \frac{4\bar{\tau}_T + \omega_T}{2m_T^*}$$

Hence

$$\text{conv}(I)(x) = \begin{cases} I'(x_0)(m_T^* + x) & \text{if } \; x \in [-m_T^*, x_0] \\ I(x) & \text{otherwise} \end{cases}$$

The convex conjugate of $\text{conv}(I)$ is

$$\bar{\Phi}(t) = I^*(t) = \begin{cases} -m_T^* t & \text{if } \; t \leqslant I'(x_0) \\ m_T^* t - \beta[4\bar{\tau}_T - \delta_T/(4Tt)] & \text{if } \; t > I'(x_0) \end{cases}$$

So the B_0 defined in (3.4) is $2TI'(x_0) = (4\bar{\tau}_T + \omega_T)/(2m_T^*)$. Finally, to prove (c) and (d), note that

$$I'(x_0) = \beta B_0/2 \qquad \text{and} \qquad x_0 = m_T^* - \delta_T/B_0^2 = m(B_0)$$

Since $\bar{I} = \bar{\Phi}^* = I^{**} = \text{conv}(I)$, the $\bar{I}_B$ defined in (3.6) is in fact

$$\bar{I}_B(x) = \text{conv}(I)(x) - \frac{\beta B}{2} x + \bar{\Phi}\left(\frac{\beta B}{2}\right) \tag{4.22}$$

As in the proof of Lemma 3.1(d), it is now easy to see that the solution set $\mathscr{F}_B$ of $\bar{I}_B(x) = 0$ must be a singleton for all $B \neq B_0$ and $\mathscr{F}_{B_0} = [-m_T^*, m(B_0)]$. Since $I_B(x) > \bar{I}_B(x)$ if $x \in (-m_T^*, m(B_0))$ and $I_B(x) = \bar{I}_B(x)$ otherwise, the equation $I_B(x) = 0$ has at most one solution for all $B \neq B_0$ and at most two solutions for $B = B_0$. Therefore, it remains to verify that $I_B(m(B)) = 0$ for all $B \neq B_0$ and $I_{B_0}(-m_T^*) = I_{B_0}(m(B_0)) = 0$, which is routine.

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